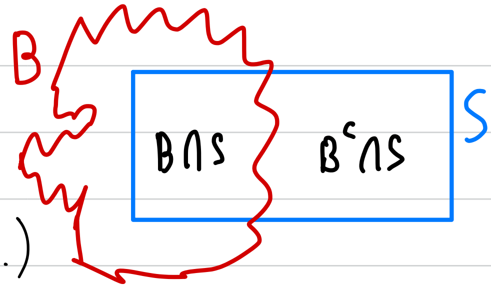
Math 564: Real analysis and measure theory
Lecture 5

Carathéodory's theorem: existence. Every premeasure μ on an algebra $\mathcal{A}$ on a set X admits an extension to a measure on the sigma-algebra $\langle \mathcal{A} \rangle_\sigma$. In fact, μ^* is such an extension.

Proof. To show that μ^* is ctbly additive on $\langle \mathcal{A} \rangle_\sigma$, it is enough to show that it is finitely additive because: outer measures are ctbly subadditive and finite additivity implies ctbl superadditivity.



Carathéodory's proof. A set B conserves a set S if

$$\mu^*(S) = \mu^*(B \cap S) + \mu^*(B^c \cap S).$$

(By subadditivity of μ^*, failure of this equality means $<$.)
Call B conservative if it conserves every set.
Let $\mathcal{M}$ denote the collection of all conservative sets. Then we prove:

(i) $\mathcal{A} \subseteq \mathcal{M}$.

(ii) $\mathcal{M}$ is a σ-algebra (hence contains $\langle \mathcal{A} \rangle_\sigma$).

(iii) μ^* is (almost by def. of $\mathcal{M}$) finitely additive on $\mathcal{M}$.

We leave the verification as HW.

Tao's proof. This proof works only for σ-finite premeasures, so assume μ is σ-finite on $\mathcal{A}$.
We will first prove it assuming μ is finite, and deduce the σ-finite case from this, HW.
We define a pseudo-metric (i.e. metric but maybe the axiom $d(x,y) = 0 \Rightarrow x = y$ doesn't hold) $d_{\mu^*} : \mathcal{P}(X) \to [0, \mu(X)]$.

$$d_{\mu^*}(A,B) := \mu^*(A \triangle B)$$

The secret of symmetric differences. $\mathcal{P}(X)$ with $\triangle$ is an abelian group, with $\emptyset$ as the identity (i.e. 0) and each element is its own inverse.

Proof. Just think of $\mathcal{P}(X)$ as 2^X, then Δ is just coordinate-wise $+$ mod 2. $\square$

Claim (a). d is a pseudo-metric.

Proof. Symmetry is by definition, as well as $d_{\mu^*}(A,A) = \mu^*(A\Delta A) = \mu^*(\emptyset) = 0$.
For the triangle inequality, let $A, B, C \subseteq X$ and observe:

$$A \Delta C = (A\Delta B) \Delta (B \Delta C) \subseteq (A\Delta B) \cup (B \Delta C),$$

so

$$d_{\mu^*}(A,C) = \mu^*(A\Delta C) \overset{\text{monot.}}{\leq} \mu^*((A\Delta B)\cup(B\Delta C)) \overset{\text{subadd.}}{\leq} \mu^*(A\Delta B) + \mu^*(B\Delta C) = d_{\mu^*}(A,B) + d_{\mu^*}(B,C). \quad \square$$

Let $\mathcal{M} := \overline{\mathcal{A}}^{d_{\mu^*}}$, the closure of $\mathcal{A}$ inside $\mathcal{P}(X)$ wrt the pseudo-metric d_{μ^*}.
We will show that $\mathcal{M}$ is a σ-algebra (hence $\mathcal{M} \supseteq \langle \mathcal{A} \rangle_\sigma$) and μ^* is finitely additive on $\mathcal{M}$.

Claim (b). The function $\begin{array}{l} \mu^* : \mathcal{P}(X) \to [0, \infty] \\ \quad\;\; A \mapsto \mu^*(A) \end{array}$ is continuous (wrt d_{μ^*}), in fact, 1-Lipschitz.

Proof. Just note that $\mu^*(A) = \mu^*(A\Delta\emptyset) = d_{\mu^*}(A,\emptyset)$, so
$$|\mu^*(A) - \mu^*(B)| = |d_{\mu^*}(A,\emptyset) - d_{\mu^*}(B,\emptyset)| \leq d_{\mu^*}(A,B)$$ by the Δ-inequality. $\square$

Claim (c). The function $\begin{array}{l} \mathcal{P}(X) \to \mathcal{P}(X) \\ \quad A \mapsto A^c \end{array}$ is continuous, in fact an isometry.

Proof: Just note that $A\Delta B = A^c \Delta B^c$, so $d_{\mu^*}(A,B) = \mu^*(A\Delta B) = \mu^*(A^c, B^c) = d_{\mu^*}(A^c, B^c)$. $\square$

This implies that $\mathcal{M}$ is closed under complements: if $M \in \mathcal{M}$, then $\exists (A_n) \subseteq \mathcal{A}$ with $\lim_{n\to\infty} A_n = M$, but by continuity, $\lim_{n\to\infty} A_n^c = M^c$, so $M^c \in \mathcal{M}$.

Claim (d). The function $\begin{array}{l} \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X) \\ \quad (A,B) \mapsto A \cup B \end{array}$ is continuous, in fact, 1-Lipschitz wrt the $d_{\mu^*} + d_{\mu^*}$ pseudo-metric on $\mathcal{P}(X)\times\mathcal{P}(X)$.

Thus, same holds for $(A,B) \mapsto A\cap B$ because it's a composition of 1-Lipschitz functions $()^c$ and $\cup$.

Proof. $d_{\mu^*}(A_1 \cup B_1, A_2 \cup B_2) = \mu^*\big((A_1 \cup B_1) \triangle (A_2 \cup B_2)\big) \overset{\text{monot.}}{\le} \mu^*\big((A_1 \triangle A_2) \cup (B_1 \triangle B_2)\big) \overset{\text{subadd.}}{\le}$
$\mu^*(A_1 \triangle A_2) + \mu^*(B_1 \triangle B_2) = d_{\mu^*}(A_1, A_2) + d_{\mu^*}(B_1, B_2)$,
where we check that $(A_1 \cup B_1) \triangle (A_2 \cup B_2) \subseteq (A_1 \triangle A_2) \cup (B_1 \triangle B_2)$. □

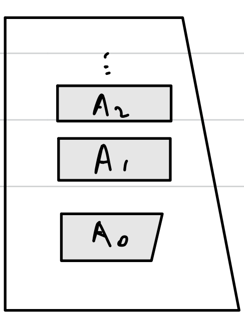
This implies that $\mathcal{M}$ is closed under finite unions, hence is an algebra: let $A, B \in \mathcal{M}$ then $\exists\, (A_n), (B_n) \subseteq \mathcal{A}$ with $\lim_n A_n = A$ and $\lim_n B_n = B$, so continuity, $\lim_n (A_n \cup B_n) = A \cup B$, hence $A \cup B \in \mathcal{M}$.

Claim (e). μ^* is finitely additive on $\mathcal{M}$.

Proof. Let $A, B \in \mathcal{M}$ which are disjoint, in order to show that $\mu^*(A \sqcup B) = \mu^*(A) + \mu^*(B)$. There are $(A_n), (B_n) \subseteq \mathcal{A}$ with $\lim_n A_n = A$ and $\lim_n B_n = B$. By the continuity of union, $\lim_n (A_n \cup B_n) = A \cup B = A \sqcup B$. By the continuity of μ^*, we have $\lim_n \mu^*(A_n) = \mu^*(A)$, $\lim_n \mu^*(B_n) = \mu(B)$, and $\lim_n \mu^*(A_n \cup B_n) = \mu^*(A \sqcup B)$. $\mu^*|_{\mathcal{A}} = \mu$, which is finitely additive, so we have: $\mu^*(A_n \cup B_n) = \mu^*(A_n) + \mu^*(B_n) - \mu^*(A_n \cap B_n)$. But by the continuity of $\cap$, $\lim_n A_n \cap B_n = A \cap B = \emptyset$, so again by the continuity of μ^*, we have $\lim_n \mu^*(A_n \cap B_n) = \mu^*(A \cap B) = \mu^*(\emptyset) = 0$, so
$$\mu^*(A \sqcup B) = \lim_n \mu^*(A_n \cup B_n)$$
$$= \lim_n \mu^*(A_n) + \lim_n \mu^*(B_n) - \lim_n \mu^*(A_n \cap B_n)$$
$$= \mu^*(A) + \mu^*(B) + 0.$$
□

Claim (f). $\mathcal{M}$ contains all ctbl unions of sets in $\mathcal{A}$.

Proof. Let $(A_n) \subseteq \mathcal{A}$, to show that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$. By disjointification, we may assume that the A_n are pairwise disjoint. It's enough to show that
$$\lim_{n \to \infty} \Big(\bigcup_{k \le n} A_k\Big) = \bigsqcup_{k \in \mathbb{N}} A_k =: A.$$



Observe $d_{\mu^*}\Big(\bigsqcup_{k \le n} A_k, A\Big) = \mu^*\Big(\bigsqcup_{k > n} A_k\Big) \overset{\text{ctbl subadd.}}{\le} \sum_{k > n} \mu^*(A_k) \to 0$ because $n \to \infty$

X, $\mu^*(X) = \mu(X) < \infty$.

the series $\sum_{k\in\mathbb{N}} \mu^*(A_k)$ converges; indeed: for all $n\in\mathbb{N}$, $\sum_{k\le n}\mu^*(A_k) = \sum_{k\le n}\mu(A_k)$

$\overset{\text{fin.add.}}{=} \mu\left(\bigsqcup_{k\le n} A_k\right) \overset{\text{monot.}}{\le} \mu(X) < \infty$, which is the only place we use the finiteness of μ. $\square$

This implies that $\mathcal{M}$ is closed under ctbl union, hence a σ-algebra.
Indeed: if $(M_n)\subseteq \mathcal{M}$, let $\varepsilon>0$ and take $A_n \in \mathcal{A}$ so $A_n \approx_{\varepsilon\cdot 2^{-(n+1)}} M_n$,
i.e. $d_{\mu^*}(A_n, M_n) \le 2^{-(n+1)}\cdot\varepsilon$. But then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$ and

$$d_{\mu^*}\left(\bigcup_n A_n, \bigcup_n M_n\right) \le \sum_{n\in\mathbb{N}} d_{\mu^*}(A_n, M_n) \le \varepsilon\cdot\sum_n 2^{-(n+1)} = \varepsilon.$$

So $\bigcup_{n\in\mathbb{N}} M_n$ is ε-close to an element of $\mathcal{M}$. But ε is arbitrary and $\mathcal{M}$ is closed, hence $\bigcup_n M_n \in \mathcal{M}$. $\square$